Optimized Network of Ground Stations for LEO Orbit Determination

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BIOGRAPHY

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ABSTRACT

While the kinematic approach of precise Orbit Determination (OD) of Low Earth Orbiter (LEO) satellites has its advantages, it may not produce a good result (even no solution) in case of bad satellite geometry or data gaps. As a way to overcome this limitation, pseudolite transceivers are proposed to improve the geometry of the satellites. For the maximum use of the pseudolites, they should be located optimally on the ground worldwide. Since an optimal network should be homogeneous and isotropic such that the errors are distributed uniformly over the network, therefore, the dispersion matrix of the estimated point coordinates will possess the Taylor-Karman structure. A Second Order Design (SOD) is used here for the network configuration, which approximates the ideal cofactor matrix by varying weights of the baselines.

Total of 63 IGS stations are used to test the optimal network design, and distances between the stations are used as measurements. An algorithm designed to find a new network station at each step is described and applied successfully in this paper. The trace of the cofactor matrix can be used as an overall measure of precision in the optimal sense. Since the trace of the cofactor matrix does not change significantly beyond 40 stations, therefore, this would be the optimal number of stations for the network. However, more than 40 stations are needed, in spite of not increasing the optimality of the network itself, because the LEO altitude is low, thus, the 40 optimal pseudolite locations will not provide sufficient data coverage.

INTRODUCTION

Kinematic Orbit Determination (OD) of LEO (Low Earth Orbiter) satellites can provide a competitive advantage over the dynamic approach which requires a dynamic force model and time consuming numerical integration processes. These advantages, however, can only be guaranteed by the quality and continuity of GPS data and the geometry between the GPS satellites and the LEO. For example, the kinematic approach is highly sensitive to a weak geometry; the positioning error can be up to three times worse than that with a good geometry. Also, the singularity due to data gaps or the lack of the GPS signal may make it impossible to achieve a solution; hence, the final solution is not continuous but being segmented according to the singular epochs (Bae et al., 2002; Kwon et al., 2003).

In order to overcome these disadvantages of the kinematic approach, pseudolite (PL) transceivers network is proposed to strengthen the constellation geometry of the positioning satellites by locating PLs onboard LEO and on the ground (Figure 1). It had been shown that the effect of the PLs on the ground significantly improves the overall geometry, especially in the height component (VDOP) (Grejner-Brzezinska et al., 2004).

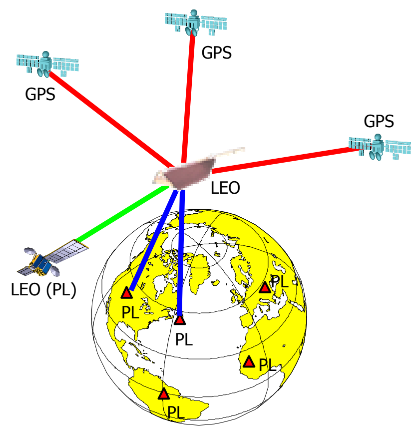

Figure 1: Constellation geometry of the positioning satellites.

One thing to be noticed here is that this approach of augmenting positioning satellites with PLs is based on tight integration between GPS and PL signals (*i.e.*, GPS and PL measurements are synchronized, controlled by the same clock). That is a major distinction of this approach from the current beacon tracking systems such as DORIS or ENVISAT, which are not integrated with GPS, or JASON which uses both beacon system and GPS, but entirely independently (Grejner-Brzezinska et al., 2004).

The issue that arises here is the optimal configuration of the ground stations where the PLs should be located. Since it can be assumed that LEO satellite orbits the Earth almost uniformly (repeat period ~15 days), the problem can be redefined as the optimal design of the PL station network independent of the position of LEO satellites. In an ideal case it is a matter of determining the position where the network stations should be. However, since most of the Earth surface is covered by water a totally uniform design is impossible, and the existing ground stations, such as IGS stations should be selected.

From this study, the optimal number of network stations and their configuration is investigated. In addition, the algorithm to improve the network is proposed, and the resulting error distribution is also discussed.

OPTIMAL DESIGN OF A NETWORK

The optimal network criteria can be composed of three components, *i.e.*, *Precision*, *Reliability* and *Costs*. Since these three components are related to each other, this problem can be handled as multi-purpose optimization problem as discussed in Schaffrin (1985, p.560):

$$\alpha_p \cdot (precision) + \alpha_r \cdot (reliability) + \alpha_c \cdot (costs)^{-1} = \max., \quad (1)$$

where α_p, α_r and α_c are chosen weight coefficients. However, this approach is not very effective because it depends on the choice of the weight coefficients. Therefore, it is commonly recommended to deal with precision, while keeping reliability and cost under control.

Grafarend (1972) proposed the requirements of the ideal network; *homogeneity* and *isotropy* of its point errors. Homogeneity means the invariance with respect to a translation in space, and isotropy represents the invariance with respect to a rotation. Under these properties, a uniform quality of the network can be achieved, resulting in circular local error ellipses of equal size. A variance-covariance matrix which is homogeneous and isotropic in an ideal network (called "criterion matrix") possesses the Taylor-Karman structure (TK-structure), which defines the covariance between two points. The TK-structure will be discussed in detail in the next section. Also, since the measurement type is not specified, this criterion matrix is independent of any linear models with specific rank-deficiency.

An optimal design problem can be considered as a process on the cofactor matrix. They can be categorized into four approaches: Zero Order Design (ZOD), First Order Design (FOD), Second Order Design (SOD), and Third Order Design (TOD) (Schaffrin, 1985, pp.556-558). ZOD optimizes the cofactor matrix by varying the datum choice based on the minimum trace of the cofactor matrix. FOD uses the varying design matrix, which is valid only in the domain of linearization. SOD determines the weights of the measurements to optimize the cofactor matrix; therefore, this will be emphasized in this paper. TOD is similar to the SOD except that the weights of additional measurements are sought for the improvement of the cofactor matrix.

TAYLOR-KARMAN STRUCTURE

The Taylor-Karman structure defines the auto- and cross-covariance between two points in the network. The distance between two points P_i and P_j is defined as

$$s := \left|\underline{r}_i - \underline{r}_j\right| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}, \quad (2)$$

where $\underline{r}_i = (x_i, y_i, z_i)^T$ and $\underline{r}_j = (x_j, y_j, z_j)^T$ represent the Cartesian coordinates of P_i and P_j, respectively. Following Grafarend (1972), the general expression for the TK-structured criterion matrix between P_i and P_j is given by

$$\sigma_0^2 C_{ij} := \begin{bmatrix} \Sigma_m(s) & 0 & 0 \\ 0 & \Sigma_m(s) & 0 \\ 0 & 0 & \Sigma_m(s) \end{bmatrix}_{3\times 3} + \left[\Sigma_l(s) - \Sigma_m(s)\right] \cdot \frac{1}{s^2} \left[(r_i - r_j)(r_i - r_j)^T\right]_{3\times 3}, \quad (3)$$

where $\Sigma_l(s)$ and $\Sigma_m(s)$ represent the longitudinal and cross-covariance functions, respectively, and σ_0^2 is the unit-free variance component. If the points P_i and P_j coincide, the two covariance functions have the same value, *i.e.,*

$$\Sigma_m(0) = \Sigma_l(0) = \sigma^2, \quad (4)$$

where σ^2 is the expected or desired variance of the estimated coordinates of the network (1 dm^2 is used here). C denotes the ideal cofactor matrix of the estimated network point coordinates.

The correlation function of an autoregressive process of the first order is given by (Schaffrin, 1985, p.586)

$$\Sigma(s)=\frac{1}{2}\left[\Sigma_m(s)+\Sigma_l(s)\right]=\sigma^2\cdot\left(\frac{s}{d}\right)\cdot K_1\left(\frac{s}{d}\right), \tag{5}$$

where K_1 is the modified Bessel function of the second kind and first order, s is the distance between two points, and d is the characteristic distance of the network, which has not yet been determined clearly. Two feasible choices for the characteristic distance are suggested:

1) d should be chosen smaller than the minimum baseline in the network (Schmitt, 1980)
2) the maximum distance of the network is an upper bound for 10d (Wimmer, 1982).

For the "potential type," additional conditions are required to distinguish the longitudinal and cross-covariance functions:

$$\Sigma_l(s)=\frac{d}{ds}\Sigma_m(s),\quad \Sigma_m(s)=\frac{2}{s^2}\int_0^s x\Sigma(x)dx\,. \tag{6}$$

In order to calculate the covariance functions, the equation (6) should be evaluated in advance, which is somewhat complicated. Instead, Grafarend and Schaffrin (1979) provided an analytical formula for the two covariance functions:

$$\Sigma_l(s)=-\frac{4d^2}{s^2}+2K_0(s/d)+\frac{4d}{s}K_1(s/d)+2\frac{s}{d}K_1(s/d)\,, \tag{7}$$

$$\Sigma_m(s)=+\frac{4d^2}{s^2}-2K_0(s/d)-\frac{4d}{s}K_1(s/d)\,, \tag{8}$$

with
K_0: as modified Bessel function of the second kind and zero order,
K_1: as modified Bessel function of the second kind and first order.

Figure 2 illustrates the longitudinal and cross-covariance functions. Covariance functions are decreasing with the distance between two points. As s/d gets closer to 10, *i.e.*, the baseline length is ten times of the characteristic distance, two points become almost decorrelated. That is the basic idea of the suggestion by Wimmer (1982).

Once the criterion matrix is computed, the weights of each measurement between stations can be computed from the SOD approach. In the SOD the difference between the cofactor matrix of the estimated point coordinates in $\hat{\xi}$, $Q_{\hat{\xi}}$, and an ideal criterion matrix (e.g., with TK-structure), C, will be minimized by varying the weights:

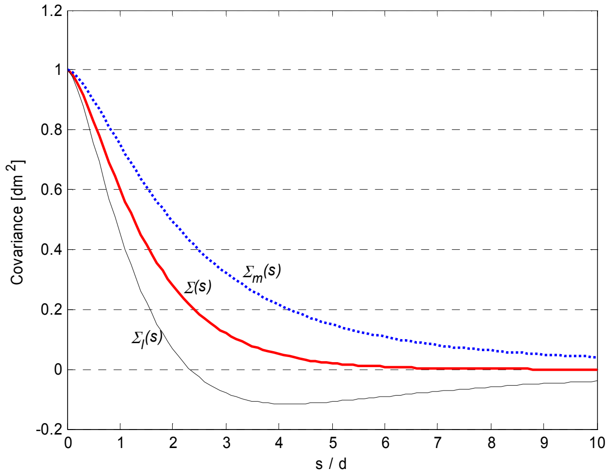

Figure 2: Longitudinal and cross-covariance functions.

$$\left\|Q_{\hat{\xi}}-C\right\|=\min\,. \tag{9}$$

Assuming uncorrelated observations in a Gauss-Markov Model, $y=A\xi+e,\ e\sim(0,\sigma_0^2P^{-1})$, this condition can be transformed into

$$\left\|A^TPA-C^{-1}\right\|=\min_{P:=diag\,\underline{p}}\,, \tag{10}$$

where C is a criterion matrix with TK-structure. Using an appropriate operator and a weighted l^2 norm, equation (10) becomes

$$\left\|(A^T\odot A^T)\underline{p}-vecC^{-1}\right\|^2_{C\otimes C}=\min_{\underline{p}=vecdiagP}\,, \tag{11}$$

where $\odot$ denotes the *Khatri-Rao product* which is defined as

$$\underbrace{A\odot B}_{nl\times m}=\left[\alpha_1\otimes\beta_1,\ \cdots,\ \alpha_m\otimes\beta_m\right], \tag{12}$$

for two matrices $A_{n\times m}=\left[\alpha_1,\ \cdots,\ \alpha_m\right]$ and $B_{l\times m}=\left[\beta_1,\ \cdots,\ \beta_m\right]$ with the same number of columns, while $\otimes$ denotes *Kronecker-Zehfuss product* defined by:

$$\underset{n\times m}{G}\otimes\underset{p\times q}{H}=[g_{ij}\cdot H]_{pn\times qm}\,. \tag{13}$$

After some manipulation of equation (11), the normal equations for the weights are given by

$$\left[(A^T\odot A^T)^T(C\otimes C)(A^T\odot A^T)\right]\underline{\hat{p}}=(A^T\odot A^T)^T\underbrace{(C\otimes C)vecC^{-1}}_{vecC}; \tag{14}$$

and finally by:

$$(ACA^T * ACA^T)\underline{\hat{p}} = vecdiag(ACA^T)\,, \tag{15}$$

where the symbol $*$ now defines the *Hadamard product* of matrices with equal size, namely:

$$\underset{k\times l}{G} * \underset{k\times l}{H} = [g_{ij} \cdot h_{ij}]_{k\times l}\,. \tag{16}$$

For more details concerning the above matrix products, see Schaffrin (1985, pp.589-590).

TEST DATA

For the application of the optimal network design technique as investigated in this paper, a total of 63 IGS tracking network stations are initially pre-selected; see http://lareg.ensg.ign.fr/ITRF/ITRF2000/results/ITRF2000_GPS.SSC. All coordinates are computed at epoch 2004.0 by applying the velocity information. As can be seen in the figure illustrating the network stations (Figure 3), the distribution of stations is denser in North America and Western Europe, while fewer stations are available in Africa and South America, which is due to the actual distribution of IGS stations.

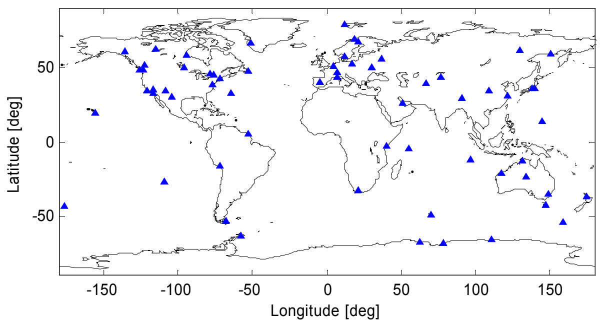

Figure 3: All IGS network stations used (63 stations).

It should be mentioned here that distance measurements are used as observables. Although any kind of measurements (direction, distance, distance ratio or angular observations, etc.) can be used, only distance measurements are introduced for the optimal network design here because, in fact, GPS observables are used for the LEO orbit determination process. Without the loss of generality, we can use different types of measurements for a different purpose of network design.

One assumption is that only the measurements between one additional candidate station and the already selected network stations are considered, as only one station is added at a time. Once a station is added to the network, the set of stations is optimal for that number of case relative to the previous case.

SEQUENTIAL ALGORITHM TO FIND OPTIMAL NETWORK STATIONS

In order to find optimal stations from the candidate group, the algorithm starts from one station and keeps adding one station at a time, which satisfies the optimality criterion at each selection step. The first station could be anywhere in the network, and the final set of stations will be different, depending on the first station. However, the resulting set should always satisfy the requirements, *i.e.*, be approximately homogeneous and isotropic. The station selection procedure can be summarized as follows (see Figure 4):

1) Choose one station from the stations list as the 1st station of the network. Although different choices of the 1st station will provide different resulting sets, the error situation is expected to be approximately uniform.

2) Find the 2nd station, which has the maximum distance from the 1st station. This maximum distance will serve as an upper limit of the characteristic distance for the computation of the covariance functions. This also gives an optimal network for the case of two stations.

3) Build a criterion matrix for two network stations. This criterion matrix will be used in the next step of the selection process. It should be mentioned that a criterion matrix will be almost the identity if the characteristic distance is smaller than the minimum baseline in the network, because the minimum baseline is quite small as compared to other baselines. Therefore, it would be better to use one tenth of the maximum baseline as an upper limit of the characteristic distance according to Wimmer (1982).

4) Compute the weights to each network station for all candidate stations. This can be done using equation (15). The criterion matrix is continuously accumulated up to the last network station. For example, for the j^{th} network station, the $3(j-1)\times 3(j-1)$ criterion matrix will be reused for the $3j\times 3j$ criterion matrix of the current candidate station. The number of rows of the design matrix will be the same as the number of network stations because no further measurements between existing network stations are assumed.

5) Find the station that has the most uniform weights to all network stations by computing the Root Mean Square (RMS) deviations of weights. Since the weight of a short baseline is larger than that of a long baseline, and the stations need to be as wide-spread as possible for the optimal configuration, the station which has the minimum RMS deviation of weights is chosen as the next network station. If a station is added to the network, it is removed from the candidate group.

6) Save the criterion matrix up to the newly chosen network station and update the maximum and characteristic distance in the network for the next step of the selection process.

7) Repeat steps 4-6 until the desired number of stations is reached.

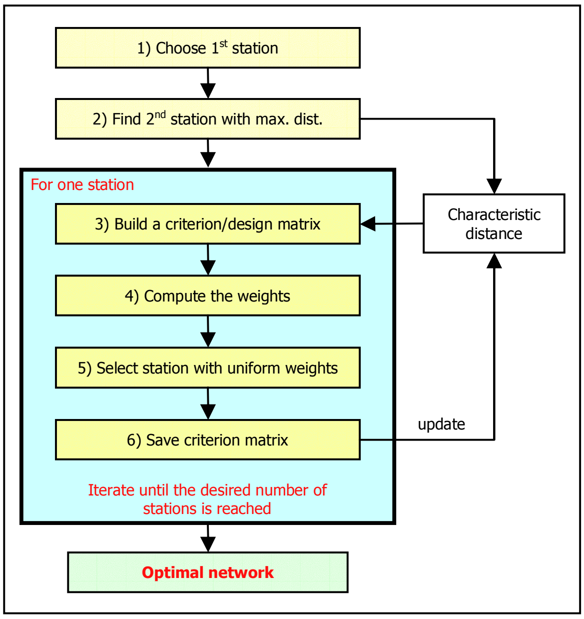

Figure 4: Flow chart of the algorithm.

THE OPTIMIZED NETWORK

Figure 5 shows the result of the optimized network (a total of 42 stations in this case). The numbers in the figure represent the sequence in the selection process. The first chosen station is ALBH in North America, and the 2nd one, KERG, is on the opposite hemisphere to station 1. Although a large number of stations are concentrated in North America and Western Europe, few of them are chosen in the final result to generate grater uniformity.

In order to test the stability of the algorithm, two cases with different starting stations are tested (Figure 6a-b). Case 1 uses ALBH in North America as a starting station, while Case 2 starts with YAKZ in the Far East (notice the circle in the figures). As can be seen in the figures, both Case 1 and 2 generate network stations almost uniformly, and more than 83% of stations (35 out of 42 stations) coincide.

Since the goal of the optimal network design is to make the cofactor matrix of the estimated point coordinates, $Q_{\hat{\xi}}$, as close to the criterion matrix as possible, the cofactor matrix should be computed using the estimated weights

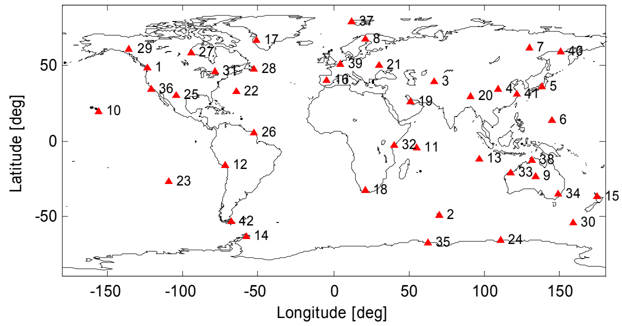

Figure 5: Selected stations using the optimal selection process (42 stations).

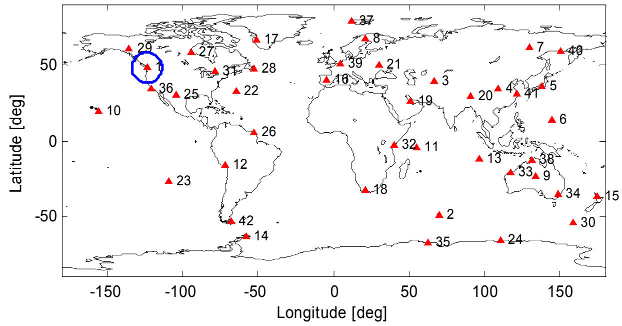

Figure 6a: Selected stations of Case 1 (starts from North America).

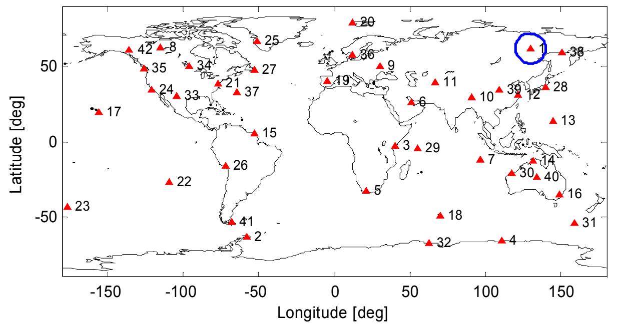

Figure 6b: Selected stations of Case 2 (starts from the Far East).

for each baseline. However, in spite of numerous advantages of the Gauss-Markov Model for geodetic networks, such as easily programmable computer calculation, the covariance matrix of the parameter estimates as a byproduct and easily understandable results (Caspary, 1987, p.9), the conventional geodetic networks experience certain datum deficiencies. In 3-D networks with distance measurements, the rank deficiency is 6 (3 translations and 3 rotations). Therefore, the resulting normal matrix has also a rank deficiency and the standard matrix inversion is not possible. Instead of a standard inversion, *the pseudoinverse* can be used for the computation of the cofactor matrix, for instance by using the Singular Value Decomposition (SVD) method (Strang et al., 1997).

Figures 7a-b show the measure of optimality. Figure 7a represents the scaled trace of the cofactor matrix and Figure 7b is the derivative of Figure 7a. It should be mentioned that for this the trace of the cofactor matrix is averaged by the number of nonzero eigenvalues. As can be seen in Figure 7b, there is no significant improvement beyond 40 stations. However, the LEO altitude (400-800 km) is very low as compared to the radius of the Earth; only half of the LEO trajectory would receive some signals from the ground-based pseudolites if 40 stations are used (Grejner-Brzezinska et al., 2004). Therefore, more PLs should be located even if there is no significant improvement in network precision anymore.

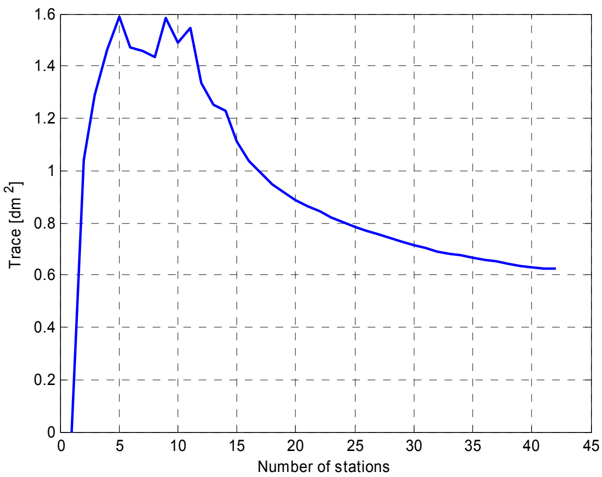

Figure 7a: Relative trace of the cofactor matrix.

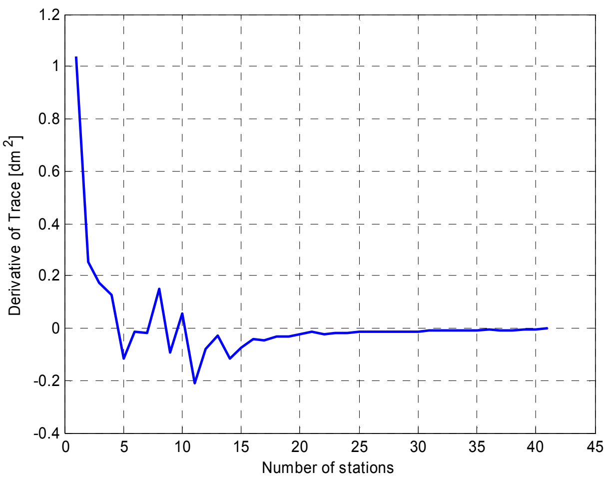

Figure 7b: Derivative of the relative trace of the cofactor matrix.

Figures 8a-b illustrate the error ellipses in the local plane. To compute the error ellipses in the local plane, the cofactor matrix in the Cartesian coordinate system should be transformed into the local plane using the relation given by (Hofmann-Wellenhof et al., 2003)

$$(Q_{\hat{\xi}})_{ENU} = R(Q_{\hat{\xi}})_{XYZ} R^T , \tag{17}$$

where R is a rotation matrix from the Cartesian to the Local-level system (East-North-Up), see Leick (1995):

$$[e,n,u]^T = R \cdot [X,Y,Z]^T . \tag{18}$$

The figures 8a-b are plotted with the same scale in both latitude and longitude directions; thus the requirements of the optimal network, *i.e.*, homogeneity and isotropy, can be clearly seen.

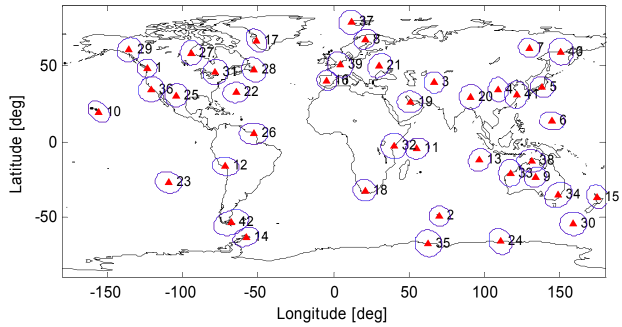

Figure 8a: Error ellipses of the optimal network (42 stations).

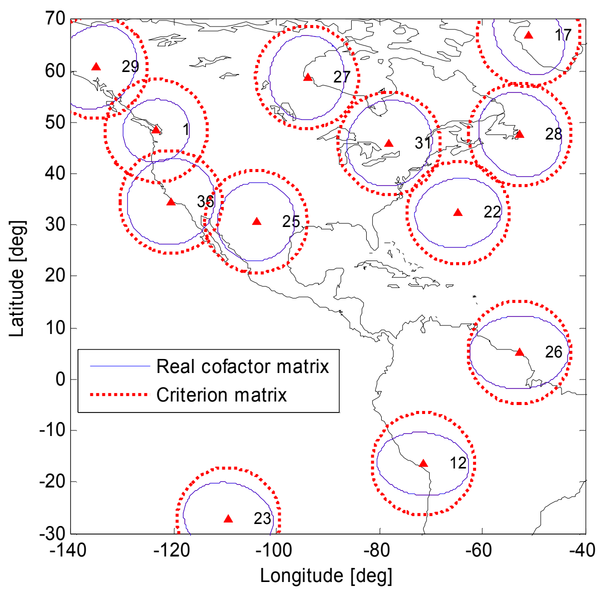

Figure 8b: Error ellipses of the optimal network (magnified in North America).

Table 1 explains how well the result follows the requirements of the optimal network. λ_{max} and λ_{min} represent the semi-major and semi-minor axes of the error ellipses, respectively, the square of which correspond to the maximum and minimum eigenvalues. The difference of maximum and minimum value in each direction is on average less than 50%. Also, the ratio of semi-major and semi-minor axis has a mean value of 1.1870, which means that they create almost circular ellipses. Even the largest value of the ratio is less than 1.5291.

Table 1: Statistics for error ellipses

	λ_{max} [dm]	λ_{min} [dm]	$\lambda_{max} - \lambda_{min}$ [dm]	$\lambda_{max} / \lambda_{min}$
Mean	0.8372	0.7095	0.1277	1.1870
RMS	0.0734	0.0728	0.0728	0.1194
Max.	0.9683	0.8706	0.3107	1.5291
Min.	0.6629	0.5872	0.0183	1.0265

*Note: RMS means Root Mean Square deviation.

Figure 9 illustrates the eigenvalues of the criterion matrix and the real cofactor matrix in the decreasing order. Since the estimates and the corresponding variance-covariance matrix are not invariant quantities of the model, as they depend on the geodetic datum choice, the eigenvalues of the cofactor matrix can be used as a measure of accuracy in certain (hyper-)dimensions defined by the corresponding eigenvectors.

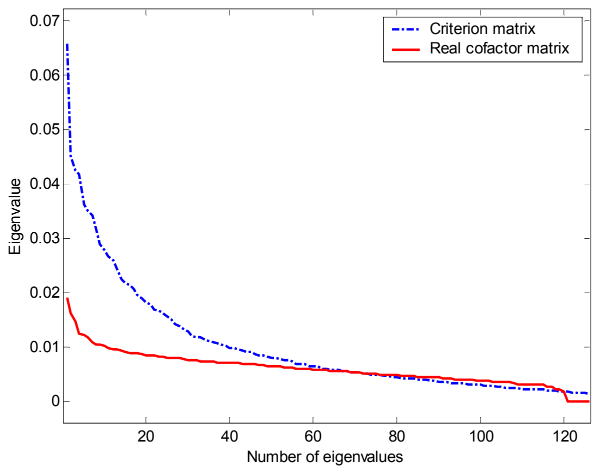

Figure 9: Eigenvalues of criterion and real cofactor matrix.

SUMMARY AND CONCLUSIONS

The analysis of an optimal network for the augmentation of positioning satellites (*i.e.*, ground PLs) was presented in this paper. The ideal variance-covariance matrix which is homogeneous and isotropic possesses the Taylor-Karman structure, which gives an enormous advantage in the network design. As a result, the optimal set of stations is almost uniformly distributed around the world, although it will have a different resulting set depending on the starting station. Clearly, the resulting error ellipses have a similar size and almost circular shape. Also, the cofactor matrix computed using the design matrix and the estimated weights of the measurements plays an important role in the overall measure of precision.

Although the optimal number of stations for the network itself is around 40, it is required to include more pseudolites on the ground to strengthen the constellation geometry because the LEO satellite altitude is low. With a satisfactory implementation of this approach, more efficient augmentation of the ground stations up to a desired number is possible without any overlapping in some areas.

In this study, only one station is considered to be added at a time. However, the algorithm can be easily modified to add a few stations at a time for efficiency. In addition, more appropriate testing quantities for the improvement of the optimality need to be investigated, besides ways to monitor the reliability. The next step in the research would be to test the selected network for observability at various LEO altitudes.

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